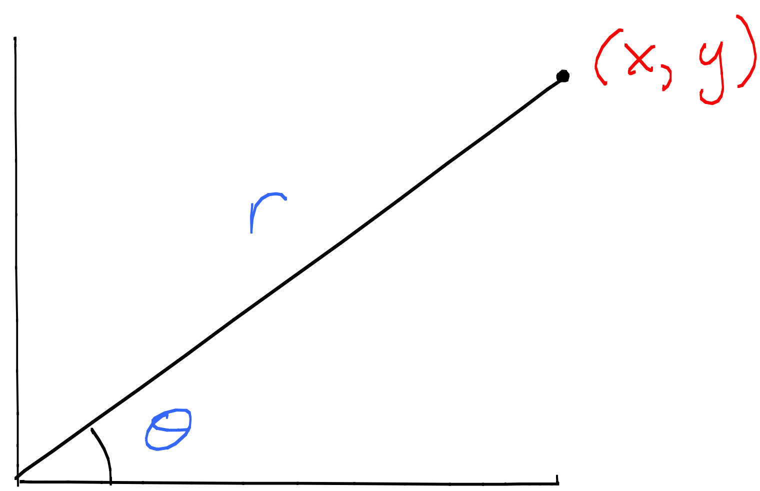
Complex Numbers: A Geometric Approach

Note Title 16/09/2008

Using complex numbers requires an initial effort, but pays off in simplifying some real computations. Many applications cannot be done without them: dynamics, circuits, control theory, and many more.

We begin with a picture in two dimensions:



A point (x,y) has polar coordinates

$r \angle \theta$

"r angle θ"

Infinitely many! $r \angle \theta + 2\pi$, $r \angle \theta - 2\pi$, $r \angle \theta + 4\pi$, ... all describe same pt.

Given two such points

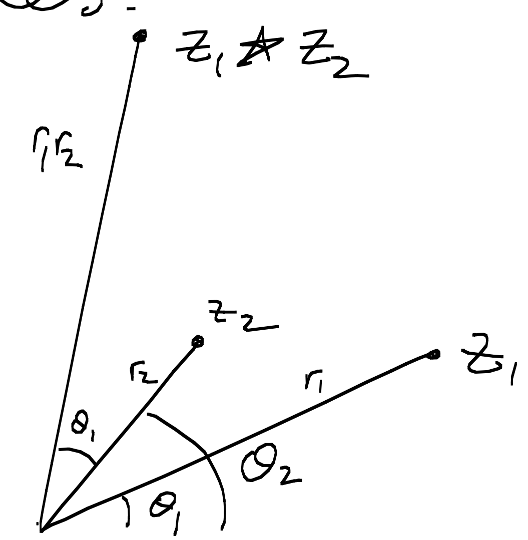
$z_1 = r_1 \angle \theta_1$ and

$z_2 = r_2 \angle \theta_2$

we define a "multiplication"

$$Z_1 \star Z_2 := r_1 r_2 \angle \theta_1 + \theta_2$$

That is, we multiply the "moduli" (one modulus, two moduli) and add the angles.



An odd thing to do.

This plays a fundamental role in many fields. See, e.g., Richard Feynmann's book QED: the strange theory of light and matter.

Exercise: Show $z_1 \star z_2 = z_2 \star z_1$, and $(z_1 \star z_2) \star z_3 = z_1 \star (z_2 \star z_3)$.

Remarks

① $1 \angle 0 \star z_1 = 1 \cdot r_1 \angle 0 + \theta$

$= r_1 \angle \theta = z_1$.

$\therefore$ $1 \angle 0$ acts like the real 1.

② If $z = 0 \angle$ anything,

$z \star z_2 = 0 \angle$ something

so $0 \angle \theta$ acts like the real 0.

(3) If $z_1 = x_1 \angle 0$ and

$z_2 = x_2 \angle 0$

then $z_1 \star z_2 = x_1 x_2 \angle 0$

so for positive real numbers x_i,

$\star$ is like real multiplication.

(4) If $z_1 = x_1 \angle \pi$

$z_2 = x_2 \angle \pi$

then $z_1 \star z_2 = x_1 x_2 \angle 2\pi$

indistinguishable from $x_1 x_2 \angle 0$

$\therefore$ things with angle π

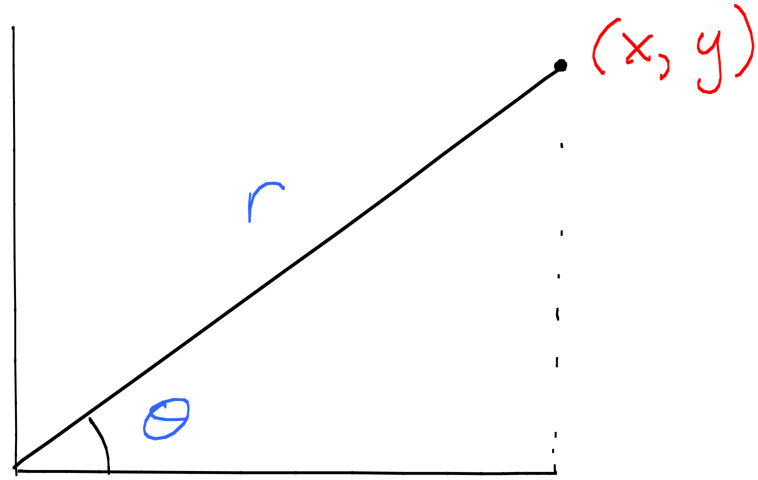
multiply like negative reals

(5) $1 \angle \pi/2 \star 1 \angle \pi/2 = 1 \angle \pi$

call this i. $i \star i =$ "minus one".

Exercise: Find a formula for $z_1 \div z_2$.

Conversion to rectangular (Cartesian) coordinates



$$y = r \sin\theta$$

$$x = r \cos\theta$$

Examples:

$$1 \angle \pi = (-1, 0)$$

$$\sqrt{2} \angle \pi/4 = (1, 1)$$

$$1 \angle \pi/2 = (0, 1) = i$$

$$1 \angle -\pi/2 = (0, -1) = -i$$

$$2 \angle \pi/3 = (1, \sqrt{3})$$

$$2 \angle -\pi/3 = (1, -\sqrt{3})$$

Addition

$$z_1 \oplus z_2 := (x_1+x_2, y_1+y_2)$$

in Cartesian coordinates.
(ugly in polar).

<u>Notation</u> $\quad i = (0,1) = 1\angle\pi/2$

$$(x, y) = (x, 0) + (0, y)$$
$$= (x, 0) + (0, 1) \star (y, 0)$$

we now <u>identify</u> $(x, 0)$ with x
$(y, 0)$ with y

so $(x, y) = x + iy$.

Exercise: Show $z_1 \star (z_2 + z_3)$
$$= z_1 \star z_2 + z_1 \star z_3.$$

(wait till after next page)

What does the multiplication rule look like in rectangular coordinates?

$$z_1 = r_1 \angle \theta_1 = (x_1, y_1)$$
$$= (r_1 \cos\theta_1, r_1 \sin\theta_1)$$

$$z_2 = r_2 \angle \theta_2 = (x_2, y_2)$$
$$= (r_2 \cos\theta_2, r_2 \sin\theta_2)$$

$$z_1 \star z_2 = r_1 r_2 \angle \theta_1 + \theta_2$$

$$= (r_1 r_2 \cos(\theta_1 + \theta_2), r_1 r_2 \sin(\theta_1 + \theta_2))$$

$$= (r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2),$$
$$r_1 r_2 (\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1))$$

$$= ((r_1 \cos\theta_1)(r_2 \cos\theta_2) - (r_1 \sin\theta_1)(r_2 \sin\theta_2),$$
$$(r_1 \sin\theta_1)(r_2 \cos\theta_2) + (r_1 \cos\theta_1)(r_2 \sin\theta_2))$$

$$= (x_1 x_2 - y_1 y_2, x_2 y_1 + x_1 y_2)$$

Exercise: for integers n, $z_1^n = r_1^n \angle n\theta$

Fractional Powers.

Take $z = 8 \angle \pi = (-8, 0)$

What is its cube root?

$$z^{1/3} = 8^{1/3} \angle \pi/3 = 2 \angle \pi/3$$
$$= (1, \sqrt{3})$$

may be surprising. But wait, there's more!

$$z = 8 \angle \pi = 8 \angle \pi + 2\pi = 8 \angle 3\pi$$

so $z^{1/3} = 8^{1/3} \angle 3\pi/3 = 2 \angle \pi$
$$= (-2, 0)$$

which might have been expected.

more yet: $z = 8 \angle \pi = 8 \angle \pi + 4\pi$

so $z^{1/3} = 8^{1/3} \angle 5\pi/3 = 2 \angle 5\pi/3$
$$= (1, -\sqrt{3}).$$

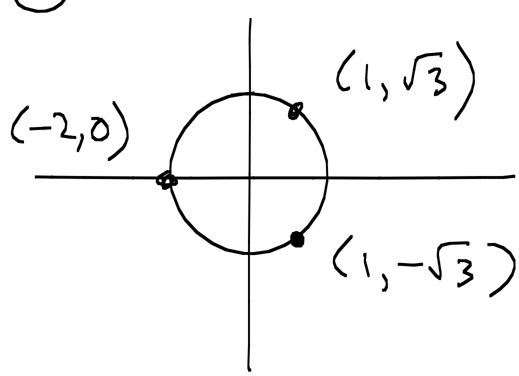
Check

$(1, \sqrt{3}) * (1, \sqrt{3}) * (1, \sqrt{3})$

$= (-2, 2\sqrt{3}) * (1, \sqrt{3}) = (-8, 2\sqrt{3} - 2\sqrt{3})$

$= (-8, 0)$ ✓

Exercise: show that other choices of k in $z = 8 \angle \pi + 2\pi k$ (k integer) lead only to these three cube roots.



Trig identities

$z_1 = 1 \angle \theta = (\cos\theta, \sin\theta)$

$z_1^3 = 1 \angle 3\theta = (\cos 3\theta, \sin 3\theta)$

What happens to $(\cos\theta, \sin\theta)^3$ using the Cartesian rule?

$$(\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\cdot(i\sin\theta) + 3\cos\theta\cdot(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

$\therefore$ comparing the two,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= 4\cos^3\theta - 3\cos\theta$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Triple-angle identities.

Euler showed $1 \angle \theta = e^{i\theta}$.

That is,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

"Euler's formula".

(need either derivatives or power series to do this).

$\theta = \pi$:

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= \underline{-1}$$

$$\therefore \quad e^{i\pi} + 1 = 0$$

"A lot of work for nothing"